CLUSTER SETS IN THE LARGE OF MEROMORPHIC FUNCTIONS ON RIEMANN SURFACES

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ABSTRACT

In this note we obtain the relationships in the large between cluster sets, asymptotic values, exceptional values of meromorphic functions on hyperbolic Riemann surfaces. Some of these are generalizations of the boundary theorems given by E. F. Collingwood and M. L. Cartwright.

Let $w = f(a)$ be a nonconstant meromorphic function on a hyperbolic Riemann surface R, and let R^* denote the Martin compactification of R. Let $\{G_{e,n}\}$ be a determinant sequence of a Kerékjártó–Stoïlow's ideal boundary point e of R, and put $\Delta_e^* = \bigcap_n \bar{G}_{e,n}$, where the bar denotes closure with respect to R^*.

In this note we assume that Δ_e^* is of positive harmonic measure.

The cluster set of $f(a)$ at e is defined as $C(f, e) = \bigcap_n \overline{f(G_{e,n})}$, where the bar denotes closure with respect to the w-sphere S. The range set of $f(a)$ at e is defined as $R(f, e) = \bigcap_n f(G_{e,n})$. A path $a = L(t)$ $(0 \leqq t < 1)$ in R is said to tend toward e, if for any n there is a $t(n)$ such that $L(t) \subset G_{e,n}$ for all $t \geqq t(n)$. If a set $A \subset R$ meets every $G_{e,n}$, then we call e an accumulation point of A. Let $\Delta_{e,n}$ denote the set of all Kerékjártó–Stoïlow's ideal boundary points of $G_{e,n}$. Let $X(f, \Delta_{e,n})$ denote the set of asymptotic values of $f(a)$ along all paths which tend toward points of $\Delta_{e,n}$. We define the two sets $X(f, e) = \bigcap_n X(f, \Delta_{e,n})$ and $X^*(f, e) = \bigcap_n \overline{X(f, \Delta_{e,n})}$. We say that $w \in X_*(f, e)$, if for any n and for every neighborhood U of a point $w \in S$, $X(f, \Delta_{e,n}) \cap U$ is of positive linear measure. $X_*(f, e)$ is closed.

Let ∂A^* denote the relative boundary of a set $A^* \subset R^*$ or S with respect to R or S. Let A^{*c} and int A^* denote the complement and the interior of A^* with respect to R^* or S. The following Lemma is a generalization of theorem 2 of [1].

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LEMMA. *$C(f, e)$ is a nondegenerate continuum.*

PROOF. $C(f, e)$ is nonempty and closed. Suppose that $C(f, e)$ is a single point w_0. Take an open disc V centered at w_0. There is then a $G_{e,N}$ such that $f(G_{e,N}) \subset V$. By the hypothesis that Δ_e^* is of positive harmonic measure, the restriction of f to $G_{e,N}$ has fine limits at almost all minimal points in Δ_e^* (see [3], p. 153). The set of these fine limits is of positive logarithmic capacity (see [3], p. 148). These fine limits belong to $C(f, e)$. This is a contradiction. Therefore $C(f, e)$ contains more than one point. It is easy to see from the proof of theorem 2 of [1] that $C(f, e)$ is connected.

The assertion of the Lemma is proved.

THEOREM 1. $\partial C(f, e) \subset X_*(f, e)$.

PROOF. Suppose, on the contrary, that there is a $w' \in \partial C(f, e)$ not in $X_*(f, e)$. There are then a $G_{e,N}$ and an open disc V' centered at w' such that $X(f, \Delta_{e,N}) \cap V'$ is of linear measure zero. By this and the Lemma, we can find a $w_0 \in V' \cap \partial C(f, e)$ not in $X(f, \Delta_{e,N})$. Let $V \subset V'$ be an open disc centered at w_0 such that $C(f, e) \not\subset \bar{V}$, and put $E = X(f, \Delta_{e,N}) \cap V$. Since E is of linear measure zero and since $w_0 \in \partial C(f, e)$, we can find the two sequences $\{G'_{e,k}\}$ and $\{V_k\}$ with the following properties (1), (2), (3) and (4).

(1) $\{G'_{e,k}\}$ is an infinite subsequence of $\{G_{e,n}\}$ such that $G'_{e,1} \subset G_{e,N}$ and $G'_{e,k+1} \subset G'_{e,k}$.

(2) $\{V_k\}$ is a sequence of open discs V_k centered at w_0 with radius r_k such that $V_1 = V$, $r_{k+1} < r_k$ and $r_k \to 0$ as $k \to \infty$.

(3) ∂V_k passes neither point of E nor branch point.

(4) ∂V_k meets the complement of $\overline{f(G'_{e,k})}$.

The following two possibilities can occur. One is the case where for each k, e is not an accumulation point of $\partial f^{-1}(V_k)$. Then we may assume that $G'_{e,k} \subset f^{-1}(V_k)$. Let $\{a_k\}$ be a sequence of points $a_k \in G'_{e,k}$. There is a simple arc L_k joining a_k to a_{k+1} in $G'_{e,k}$ such that $f(L_k) \subset V_k$. The path $L = \bigcup L_k$ tends toward e and has the property that $f(a) \to w_0$ as a tends toward e along L. This is a contradiction.

The other is the case where for a k^*, e is an accumulation point of $\partial f^{-1}(V_{k^*})$. By (4), $\partial f^{-1}(V_{k^*})$ contains no simple closed curve in G'_{e,k^*}. Suppose that $G'_{e,k^*} \cap \partial f^{-1}(V_{k^*})$ contains a path L which tends toward an $e' \in \Delta_{e,k^*}$. Let $z = h(a)$ denote a local parameter at each point of L. Let a' be a point of L. By (4), a function element of $z = h \circ f^{-1}(w)$ around $w' = f(a')$ can not be continued round ∂V_{k^*}. Therefore as a' tends toward e' along L, w' converges to a $w^* \in \partial V_{k^*}$. Therefore $w^* \in E$. This contradicts (3).

Thus $G'_{e,k^*} \cap \partial f^{-1}(V_{k^*})$ consists of infinitely many connected components L_n

with the properties that L_n is relatively compact on R, that L_n meets $\partial G'_{e,k\,*}$ and that e is an accumulation point of $\bigcup L_n$. However this can never occur since $\partial f^{-1}(V_{k\,*})$ is level curves (see [2], p. 137). This is also a contradiction.

The assertion of Theorem 1 is proved.

Evidently $X_*(f,e) \subset X^*(f,e) \subset C(f,e)$. Hence $\operatorname{int} X_*(f,e) \subset \operatorname{int} X^*(f,e) \subset \operatorname{int} C(f,e)$. It follows from Theorem 1 that $\partial C(f,e) \subset \partial X_*(f,e)$. It is further seen that $\partial C(f,e) \subset \partial X^*(f,e)$. The following Corollary 1 which is a generalization of theorem 10 of [1] is obtained.

Corollary 1. $\partial C(f,e) \subset \partial X_*(f,e) \cap \partial X^*(f,e)$.

The following Corollaries 2 and 3 are immediately obtained from Theorem 1.

Corollary 2. *If $X_*(f,e)$ is empty, then $C(f,e) = S$.*

Corollary 3. *If, for an n^* and for a neighborhood U of a $w_0 \in C(f,e)$, $E^* = X(f, \Delta_{e,n\,*}) \cap U$ is of linear measure zero, then $w_0 \in \operatorname{int} C(f,e)$.*

Theorem 2. *Under the hypothesis of Corollary 3, any $w^* \in R(f,e)^c \cap U(w_0)$ for a neighborhood $U(w_0) \subset C(f,e) \cap U$ of w_0 belongs to $X(f,e)$.*

Proof. There is a $G_{e,N} \subset G_{e,n\,*}$ such that $w^* \notin f(G_{e,N} \cup \partial G_{e,N})$. Take any $G_{e,n} \subset G_{e,N}$ and an open disc $V \subset U(w_0)$ centered at w^* such that $V \cap f(\partial G_{e,n})$ is empty. Let $D \subset G_{e,n}$ be a connected component of $f^{-1}(V)$ which is not relatively compact on R. We can find a radius C of V with the properties that C meets $f(D)$ and that C passes neither point of E^* nor branch point.

Let $L \subset D$ be a connected component of $f^{-1}(C)$, and let $z = h(a)$ denote a local parameter at each point of L. Let a' be a point of L. Suppose that a function element of $z = h \circ f^{-1}(w)$ around $w' = f(a')$ can be continued to $w' \neq w^*$ not to w^*, when w' goes toward w^* along C. Then L tends toward a point of $\Delta_{e,n}$. Therefore $w' \in E^*$. This is a contradiction. Therefore as a' tends toward a point of $\Delta_{e,n}$, w' converges to w^*. Thus $w^* \in X(f,e)$.

The assertion of Theorem 2 is proved.

Theorem 9 of [1] is the main theorem in the large. The following Corollary 4 which is a generalization of (ii) of theorem 9 is obtained from Theorem 2 and Corollary 2.

Corollary 4. *If $X_*(f,e)$ is empty, then $R(f,e)^c \subset X(f,e)$.*

Corollary 5. *Under the hypothesis of Corollary 3, $R(f,e)^c \cap U(w_0)$ for the $U(w_0)$ in Theorem 2 is of linear measure zero.*

Corollary 6. *If $X_*(f,e)$ is empty, then $R(f,e)^c$ is of linear measure zero.*

References

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